Oscillation Solutions of First-Order Neutral Differential Equations With Variable Coefficients

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ABSTRACT

In this paper we obtain a necessary and sufficient conditions for all solutions of the first-order neutral delay differential equation

\[
\frac{d}{dt} \left[ y(t) + p(t)y(t - \tau) \right] + Q(t)y(t - \sigma) = 0.
\]

Key Words: Continuous function; Neutral delay; Weakly and strongly oscillatory

INTRODUCTION

A neutral delay differential equation is a differential equation in which the highest-order derivative of the unknown function appears in the equation both with and without delays. Our aim in this paper is to obtain sufficient conditions for the oscillation of all solutions of Eq. (1.1).

As usual a solution of Eq. (1.1) is said to be oscillate if it has arbitrarily large zeros, otherwise the solution is called non-oscillatory if it is eventually positive or eventually negative.

We will see in Lemma 2.1 (Ladas & Chuanxi, 1990) that every solution of Eq. (1.1) oscillates if and only if the derivative of every continuously differentiable solution of the associated NDDE

\[
\dot{Z}(t) + P(t - \sigma) \frac{Q(t)}{Q(t - \tau)} \dot{Z}(t - \tau) + Q(t)z(t - \sigma) = 0.
\]

In the sequel it is very important to establish sufficient conditions for the oscillation of the derivative of every continuously differentiable solution of NDDE of the form
\[ \dot{Z}(t) + R(t)Z(t) + Q(t)Z(t - \sigma) = 0 \] and we will investigate this in theorem 2.1.

Let \( m = \max\{\tau, \sigma\} \). We mean by a solution of Eq. (1.1) the function \( y \in C[\left[ t_1 - m, \infty \right)) \) for some \( t_1 \geq t_0 \) such that \( y(t) + p(t)y(t - \tau) \) is continuously differentiable on \( \left[ t_0, \infty \right) \) and implies Eq. (1.1) is satisfied for \( t \geq t_1 \).

Suppose that \( P, Q \in C[\left[ t_0, \infty \right)), \quad \tau, \sigma \in R^+, \quad t_1 \geq t_0 \) and assume \( \phi \in C[\left[ t_1 - m, t_1 \right), R] \) be a given initial function, then we can show by the method of steps that Eq. (1.1) has a unique solution \( y \in C[\left[ t_1 - m, \infty \right), R] \), implies \( y(t) = \phi(t) \) for \( t \in \left[ t_1 - m, t_1 \right] \).

Now we will use the following definition in this paper.

Definition:

A continuous function \( x(t) \) is said to be (weakly) oscillatory if there exists a sequence \( \{t_k\} \) such that \( t_k \to \infty \) as \( k \to \infty \) and \( x(t_k) = 0 \), for \( k \geq 1 \).

A continuous function \( x(t) \) is said to be (strongly) oscillatory if there exist two sequences \( \{t_k\} \) and \( \{t_{k'}\} \) which tend to infinity as \( k \to \infty \) and implies \( x(t_k) < 0 < x(t_{k'}) \), \( k \geq 1 \) (Ladas & Chuanxi, 1990).

Clearly if a solution strongly oscillatory then it is also oscillatory (Ladas, Lakshmikantham, & Zhang, 1987).

**SUFFICIENT CONDITIONS FOR OSCILLATIONS**

The following lemma from (Ladas, et al,1987) will be useful to obtain new sufficient conditions for the oscillation of all solution of eq. (1.1).

**Lemma 2.1.** Assume that (1.2) is satisfied. Then every solution of eq. (1.1) oscillates if and only if the derivative of every continuously differentiable solution of eq.(1.4) oscillates.

**Proof.** Let the function \( y(t) \) be an eventually solution of eq. (1.1) and set \( z(t) = y(t) + p(t)y(t - \tau) \).

Then from (1.1) implies

\[ \dot{z}(t) = -Q(t)y(t - \sigma) \leq 0. \] (2.1)

Then the hypothesis that \( Q(t) > 0 \) implies that \( \dot{z}(t) \) is non-oscillatory.
It is easy to see that if \( z(t) \) is a continuously differentiable solution of eq. (1.4), then \( \dot{z}(t) \) is non-oscillatory. The following theorem from (Ladas, et al., 1987) is useful to obtaining a sufficient condition of strong oscillation solution of eq. (1.4).

**Theorem 2.1.** Assume that

\[
R \in C^1\left([t_0, \infty)\right), \ Q \in C\left([t_0, \infty), R^+\right), \ \tau, \sigma \in R^+.
\]

![Equation Image](image)

Then the derivative of every continuously differentiable non-trivial solution of eq.(1.3) is strongly oscillatory.

**Proof.** Let for the proof of contradiction that eq.(1.3) has continuously differentiable non-trivial solution \( z(t) \), such that eventually

\[
\dot{z}(t) \leq 0.
\]

If eventually \( \dot{z}(t) = 0 \), then \( z(t) = C \neq 0 \) and contradicts of (2.3). Implies \( \dot{z}(t) \neq 0 \) and \( z(t) \) is eventually positive or negative.

According to (2.4) we can see that eventually either

\[
R(t) > -1.
\]

or

\[
R(t) < -1.
\]

Assume

\[
W(t) = Z(t) + R(t)Z(t-\tau).
\]

Then from (1.4)

\[
\dot{W}(t) = \dot{R}(t)Z(t-\tau) - Q(t)Z(t-\sigma).
\]
We suppose that (2.7) holds. If $z(t)$ is eventually negative, then from (2.6) and (2.9) we have
\[ W(t) \leq (1 + R(t))Z(t - \tau) < 0 \] (2.11)
and (2.10) we get
\[ \dot{W}(t) \geq -Q(t)Z(t - \sigma). \] (2.12)

Then integrating (2.12) from $t_1$ to $t$ with $t_1$ sufficiently large we obtain
\[ W(t) - W(t_1) \geq \int_{t_1}^{t} Q(s)Z(s - \sigma)ds \geq -Z(t_1 - \sigma)\int_{t_1}^{t} Q(s)ds, \text{ for } t \geq t_1, \]
which according to (2.3) leads to the fact that
\[ \lim_{t \to \infty} w(t) = \infty. \]

Which contradicts (2.11), so $z(t)$ must be positive eventually. Then by (2.6)
\[ \lim_{t \to \infty} z(t) = L, \text{ for } 0 \leq L \leq \infty \text{ exists.} \] (2.13)

Since from (2.10) and (2.3)
\[ W(t) \leq 0 \text{ and } W(t) \neq 0. \]

From (2.9) using (1.1) we have
\[ W(t) \geq z(t) - z(t - \tau). \] (2.15)

Now by taking the limits of (2.15) as $t \to \infty$, we obtain
\[ \lim_{t \to \infty} w(t) \geq 0. \] (2.16)

Which according to (2.14), we get
\[ w(t) > 0. \] (2.17)

When
\[ w(t) \leq [1 + R(t)]z(t - \tau) \text{ and } 1 + R(t) > 0, \]
we have
\[ \frac{Q(t)}{1 + R(t + \tau - \sigma)} w(t + \tau - \sigma) \leq Q(t)Z(t - \sigma) = \tilde{R}(t)z(t - \tau) - \dot{w}(t) \]
and then
\[ \dot{w}(t) + \frac{Q(t)}{1 + R(t + \tau - \sigma)} w(t - (\sigma - \tau)) \leq \tilde{R}(t)z(t - \tau) \leq 0. \] (2.18)

We assume (2.8) holds. If $z(t)$ eventually positive, then according to (2.6), we see that (2.13) holds. Also (2.2), (2.3), (2.4) and (2.10) imply that (2.14) holds.
From (1.3), (2.6) and (2.8) we implies that
\[ z'(t) - z'(t - \tau) + Q(t)z(t - \sigma) \leq 0. \tag{2.19} \]

Let \( v(t) = z(t) - z(t - \tau). \tag{2.20} \)

Then from (2.13), (2.19) and (2.3) we get
\[
\lim_{t \to \infty} v(t) = 0, \quad v'(t) \leq -Q(t)z(t - \sigma) \leq 0 \quad \text{and} \quad v(t) \neq 0.
\]

Implies
\[ v(t) > 0. \tag{2.21} \]

Therefore from (2.20) and (2.21) we get \( z(t) > z(t - \tau). \) This contradicts (2.6) and then \( z(t) \) must be eventually negative. Implies (2.10) gives
\[ \dot{W}(t) \geq Q(t)z(t - \tau). \tag{2.22} \]

Which implies \( -z(t) \) is an increasing positive function and (2.3) holds, then by integrating (2.22) from \( t_i \) to \( t \) and take \( t \to \infty. \)

\[ \lim_{t \to \infty} w(t) = \infty. \]

We find that
\[ W(t) = z(t) + R(t)z(t - \tau) \leq [1 + R(t)]z(t - \tau). \tag{2.23} \]

Then
\[
\frac{Q(t)}{1 + R(t)(t - \tau - \sigma)} W(t + \tau - \sigma) \geq Q(t)z(t - \sigma) = \dot{R}(t)z(t - \tau) - \dot{w}(t).
\]

Implies
\[ \dot{w}(t) - \frac{Q(t)}{1 + R(t + (t - \sigma))} w(t + (t - \sigma)) \geq \dot{R}(t)z(t - \sigma) \geq 0. \tag{2.24} \]

If we see that from (2.8) and (2.5) we get \( \sigma < \tau, \) see (Onose, 1993), then if (2.5) holds and \( \sigma < \tau, \) then the inequality (2.24) does not an eventually positive solution. Implies contradicts our assumption (2.23) and the proof of theorem is complete.

Now apply Lemma 1.1 and Theorem 2.1 to establish new sufficient conditions for the oscillation of all solution of eq. (1.1). In particular one of our results includes a recent result for eq. (1.1).
For convenience, we set
\[ R(t) = P(t - \sigma) \frac{Q(t)}{Q(t - \tau)}. \]

**Theorem I.** Assume that
\[ P \in C^1\left([t_0, \infty), R^+\right], \ Q \in C^1\left([t_0, \infty), R^+\right], \ \tau, \sigma \in R^+ , \quad (2.25) \]
\[ P(t) > 0 \quad \text{and} \quad \dot{R}(t) \leq 0, \quad (2.26) \]
and
\[ \liminf_{t \to \infty} \int_{t-\tau}^{t-\sigma} \frac{Q(s)Q(s - \sigma)}{Q(s - \sigma) + P(s + \tau - 2\sigma)Q(s + \tau - \sigma)} \, ds > \frac{1}{e}. \quad (2.27) \]

Then every solution of eq. (1.1) oscillates (Chuanxi & Ladas, 1991).

**Theorem II [1].** Assume that (2.3), (2.25) and (2.27) hold and either
\[ P(t) \leq -1 \quad \text{and} \quad \dot{R}(t) \leq 0 \quad \text{for} \quad t \geq t_0 \]
\[ \text{or} \]
\[ -1 \leq P(t) \leq 0, \quad -1 < R(t) \quad \text{and} \quad \dot{R}(t) \leq 0 \quad \text{for} \quad t \geq t_0. \quad (2.29) \]

Then every solution of eq. (1.1) oscillates (Chuanxi & Ladas, 1989).

**Theorem III.** Assume that
\[ P \in R^{[1, -1]}, \ Q \in C\left([t_0, \infty), R^+\right], \tau, \sigma \in R^+, \quad (2.30) \]
\[ Q(t) \text{ is } \tau \text{-periodic} \quad (2.31) \]
and let
\[ \frac{1}{p+1} \left[ \liminf_{t \to \infty} \int_{t-\sigma}^{t-\tau} Q(s) \, ds \right] > \frac{1}{e}. \quad (2.32) \]

Then every solution of eq. (1.1) oscillates.

**Theorem 2.2.** Assume that
\[ P \in C^1\left([t_0, \infty), R^+\right], \ Q \in C\left([t_0, \infty), R^+\right], \quad (2.33) \]
\[ \dot{P}(t) \leq 0 \quad \text{and} \quad P(t) \neq -1 \quad \text{for} \quad t \geq t_0 \quad (2.34) \]
and if \( Q(t) \) is \( \tau \text{-periodic} \), let
\[ \liminf_{t \to \infty} \int_{t-\sigma}^{t-\tau} \frac{Q(s)}{1 + P(s + \tau - 2\sigma)} \, ds > \frac{1}{e}. \quad (2.35) \]
Then every solution of eq. (1.1) oscillates (Chuanxi, et al., 1989).

**Example 1.** The neutral delay differential equation

\[
\frac{d}{dt} \left[ y(t) - \frac{t(2 - \cos t)}{(t - \pi)(2 - \sin t)} y(t - \pi/2) \right] + (2 + \sin t/y(t - \pi) = 0; \quad t \geq 1
\]

satisfies the hypothesis of theorem 2.2, then every solution of this equation oscillates.

**Example 2.** The neutral delay differential equation

\[
\frac{d}{dt} \left[ y(t) - \left( 2 - \frac{1}{t+1} \right) y(t - \pi) \right] + (1 + \cos 2t) y(t - 1) = 0; \quad t \geq 0,
\]

satisfies the hypothesis of theorem 2.1, therefore every solution of the above equation oscillates.

**General remarks** (Ladas & Chuanxi, 1990):

1. It is easy to see that all the hypothesis of theorem 2.1 are satisfied. Therefore the derivative of every continuously differentiable solution of eq. (1.3) oscillates. Then by lemma 1.1 all solution of eq. (1.1) also oscillates.

2. It is easy to see that if \( p(t) \geq 0 \), then (2.27) gives (2.3).

3. When (2.3) holds and \( R(t) = -1 \), then every solution of eq. (1.1) oscillates (Chuanxi, et al., 1989).

4. If \( P = 1 \), then according to theorem III the result is also true.

5. When (1.3) hold and if \( P = -1 \), then every solution of eq. (1.1) oscillates (Laday & Sficos, 1986)

**REFERENCES**


